ON COMPLEMENTARY MANIFOLDS AND PROJECTIONS IN SPACES L_p AND l_p†

BY

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Introduction. If Λ is a Banach space, $\mathfrak{M}$ a closed linear subset of Λ, then a closed linear subset $\mathfrak{N}$ such that every $f \epsilon \Lambda$ is uniquely expressible as $g+h$, $g \epsilon \mathfrak{M}$, $h \epsilon \mathfrak{N}$, is called a complementary manifold to $\mathfrak{M}$.

In his treatise on linear operations, Banach‡ presents the following two problems ((B), pp. 244–245).

(a) *To every closed linear subset $\mathfrak{M}$ in L_p, $1<p\neq 2$, does there exist a complementary manifold?*

(b) *To every closed linear subset $\mathfrak{M}$ in l_p, $1<p\neq 2$, does there exist a complementary manifold?*

We show in this paper that the answer to both questions is "no."

In Chapter 1 of this paper we show that if a certain limit has the value ∞, then the answer is negative.§ In Chapters 2 and 3, it is proved that this is indeed the case. In the concluding section we discuss the relation of various other problems to (a) and (b).

CHAPTER 1. $C(\mathfrak{M})$ AND $\overline{C}(\Lambda)$

1.1. Let Λ denote a separable space with a p-norm, i.e., Λ is either L_p or l_p or the set of ordered n-tuples of real numbers $\{(a_1, \cdots, a_n)\}$, $l_{p,n}$, with the norm $\|(a_1, \cdots, a_n)\| = (\sum_{i=1}^{n} |a_i|^p)^{1/p}$. We also let $l_{p,\infty} = l_p$. The notation $p' = 1/(p-1)$, $1/p+1/p'=1$ will be used throughout.

Let $\mathfrak{M}$ be a closed linear manifold in Λ. Let R denote the set of real numbers $0 \leqq a \leqq \infty$, and let $r(a, b) = a/(1+a) - b/(1+b)$, $(\infty/(1+\infty) \equiv 1)$. It is easy to see that R with the metric $|r(a, b)|$ is a complete metric space and is homeomorphic to the closed interval $(0, 1)$.

If $\mathfrak{M}$ is a closed linear manifold in Λ, a limited transformation E such that $E\Lambda = \mathfrak{M}$, $E^2 = E$, is said to project Λ on $\mathfrak{M}$.

If E is a limited transformation, we denote by $|E|$ the bound of E.

LEMMA 1.1.1. *Let $\mathfrak{M}$ be a closed linear manifold in Λ. The existence of a*

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‡ *Théorie des Opérations Linéaires*, Warsaw, 1932. We shall refer to this work as (B). The quantities considered in the sequel are assumed to be real valued unless explicitly stated to the contrary.

§ This result was obtained while the author was a National Research Fellow at Brown University, Providence, R. I.

complementary manifold $\mathfrak{N}$ *to* $\mathfrak{M}$ *is equivalent to the existence of a projection* E *of* Λ *on* $\mathfrak{M}$.

Suppose $\mathfrak{N}$ exists. Let E be the transformation which is such that $f=g+h$, $g\epsilon\mathfrak{M}$, $h\epsilon\mathfrak{N}$, then $Ef=g$. Owing to the properties of $\mathfrak{N}$, E is single-valued, additive, homogeneous and defined everywhere. Now let $\{f_i\}$ be a sequence, which approaches f and such that if $f_i=g_i+h_i$, $g_i\epsilon\mathfrak{M}$, $h_i\epsilon\mathfrak{N}$ the g_i form a convergent sequence with limit g'. Then g' is $\epsilon\mathfrak{M}$ and the sequence $h_i=f_i-g_i$ also converges to a $h'\epsilon\mathfrak{N}$. By continuity we have $f=g'+h'$. The uniqueness of the resolution of f now implies that $Ef=g'$ or that E is closed. Theorem 7 of (B), Chapter III, p. 41, now implies that E is bounded. Since the range of E is included in $\mathfrak{M}$ and for every $f\epsilon\mathfrak{M}$, $Ef=f$, we see that the range of E is $\mathfrak{M}$ and $E^2=E$ or E is a projection of Λ on $\mathfrak{M}$.

Now suppose E exists. Let $\mathfrak{N}$ be the set of g's in Λ for which $Eg=0$. Since E is limited and linear, $\mathfrak{N}$ is a closed linear manifold. If $f\epsilon\Lambda$, $f=Ef+(1-E)f$ where $Ef\epsilon\mathfrak{M}$ and $(1-E)f\epsilon\mathfrak{N}$, since $E(1-E)f=(E-E^2)f=0$. On the other hand if $h\epsilon\mathfrak{M}$, $h=Ef$ for some $f\epsilon\Lambda$, and hence $Eh=E^2f=Ef=h$. Thus if $h\epsilon\mathfrak{N}\cdot\mathfrak{M}$, $0=Eh=h$, or $\mathfrak{M}\cdot\mathfrak{N}=\{0\}$. Now let f again be $\epsilon\Lambda$, $f=g+h=g'+h'$, g, $g'\epsilon\mathfrak{M}$, h, $h'\epsilon\mathfrak{N}$. Then $g-g'=h'-h$, and since $h-h'\epsilon\mathfrak{N}$, $g'-g\epsilon\mathfrak{M}$, and $\mathfrak{M}\cdot\mathfrak{N}=\{0\}$, this implies $g-g'=h'-h=0$. This shows that $f\epsilon\Lambda$ can only be expressed in one way as $h+g$, $h\epsilon\mathfrak{N}$, $g\epsilon\mathfrak{M}$.

We may therefore consider problems (a) and (b) in the following equivalent form.

(A) *To every closed linear manifold* $\mathfrak{M}$ *of* L_p, $1<p\neq 2$, *is there a projection of* L_p *on* $\mathfrak{M}$?

(B) *To every closed linear manifold* $\mathfrak{M}$ *of* l_p, $1<p\neq 2$, *is there a projection of* l_p *on* $\mathfrak{M}$?

Let $\Lambda_1, \cdots, \Lambda_n$, $n=1, 2, \cdots, \infty$, $\Lambda_\infty\equiv\Lambda$ be a set of spaces. Let $\sum_{\alpha=1}^{n}\oplus\Lambda_\alpha=\Lambda_1\oplus\cdots\oplus\Lambda_n$ denote the space of ordered sets of elements $\{f_1, f_2, \cdots, f_n\}$ ($f_\infty\equiv f$) $f_\alpha\epsilon\Lambda_\alpha$, such that $\sum_{\alpha=1}^{n}\|f_\alpha\|^p<\infty$, with a norm defined by the equation

$$\|\{f_1, f_2, \cdots, f_n\}\| = \left(\sum_{\alpha=1}^{n}\|f_\alpha\|^p\right)^{1/p}.$$

$\Lambda_\alpha\cong\Lambda_\beta$ is to mean that there exists a one-to-one isometric mapping of Λ_α on Λ_β.

LEMMA 1.1.2. (a) $\sum_{\alpha=1}^{n}\oplus l_{p,n_\alpha}=l_{p,m}$, $n_\alpha=1, 2, \cdots, \infty$, *if* $\sum_{\alpha=1}^{n}n_\alpha=m$.

(b) $\sum_{\alpha=1}^{n}\oplus\Lambda_\alpha\cong L_p$ *if* $\Lambda_\alpha=L_p$, *for each* α.

The proof of this lemma may be left to the reader.

1.2. Let $\mathfrak{M}$ be a closed linear manifold in Λ. We define a function $C(\mathfrak{M})$, which takes on values in R as follows. If there exists no projection of Λ on $\mathfrak{M}$, then $C(\mathfrak{M}) = \infty$. Otherwise $C(\mathfrak{M}) = \text{g.l.b.}\ (|E|\,;\, E\Lambda = \mathfrak{M}, E^2 = E)$. Similarly we define the function $\overline{C}(\Lambda)$ as l.u.b. $(C(\mathfrak{M}), \mathfrak{M} \subseteq \Lambda)$.

LEMMA 1.2.1. *Let Λ_1 and Λ_2 be such that Λ_1 is equivalent* [(B), p. 180] *to a closed linear manifold $\mathfrak{M}$ of Λ_2. Let $\mathfrak{M}$ be such that there exists a projection E of Λ_2 on $\mathfrak{M}$, with $|E| = 1$, $\mathfrak{N}$ the set of f's $\epsilon\Lambda$, for which $Ef = 0$. Let $\mathfrak{P}$ be any closed linear manifold of Λ_2, such that if $f \epsilon \mathfrak{P}$, then $f = g + h$, $g \epsilon \mathfrak{P}\cdot\mathfrak{M}$, $h \epsilon \mathfrak{P}\cdot\mathfrak{N}$. Let $\mathfrak{P}_1$ in Λ_1 be the manifold which corresponds to $\mathfrak{P}\cdot\mathfrak{M}$. Then $C(\mathfrak{P}_1) \leqq C(\mathfrak{P})$.*

If $C(\mathfrak{P}) = \infty$, our statement is true. Suppose $C(\mathfrak{P})$ is $< \infty$. Let F be any projection of Λ_2 on $\mathfrak{P}$. Then EF is a projection on $\mathfrak{P}\cdot\mathfrak{M}$. For if $f_1 \epsilon \Lambda_2$, $f = Ff_1 = g + h$, $g \epsilon \mathfrak{P}\cdot\mathfrak{M}$, $h \epsilon \mathfrak{P}\cdot\mathfrak{N}$, and $EFf = g$ or the range of EF is included in $\mathfrak{P}\cdot\mathfrak{M}$. Also for every $h \epsilon \mathfrak{P}\cdot\mathfrak{M}$, we have $EFh = Eh = h$. This with our previous statement shows that $(EF)^2 = EF$ and that the range of EF is exactly $\mathfrak{P}\cdot\mathfrak{M}$.

Let $(EF)'$ be EF considered only on $\mathfrak{M}$. Obviously $(EF)'$ projects $\mathfrak{M}$ on $\mathfrak{P}\cdot\mathfrak{M}$. Let G be the corresponding transformation on Λ_1. Then $C(\mathfrak{P}_1) \leqq |G| = |(EF)'| \leqq |EF| \leqq |E|\cdot|F| = |F|$ or $C(\mathfrak{P}_1) \leqq |F|$. Since F was any projection on $\mathfrak{P}$, $C(\mathfrak{P}_1) \leqq C(\mathfrak{P})$.

LEMMA 1.2.2. *If Λ_1 and Λ_2 are as in Lemma* 1.2.1, $\overline{C}(\Lambda_1) \leqq \overline{C}(\Lambda_2)$. *In particular if $\Lambda_2 = \Lambda_0 \oplus \Lambda_1$, $\overline{C}(\Lambda_1) \leqq \overline{C}(\Lambda_2)$.*

Let $\mathfrak{P}_1$ be any closed linear manifold of Λ_1, $\mathfrak{P}$ the corresponding set of elements in $\mathfrak{M}$. $\mathfrak{P}$ is a closed linear manifold satisfying the conditions given in Lemma 1.2.1, since $\mathfrak{P}\cdot\mathfrak{M} = \mathfrak{P}$, $\mathfrak{P}\cdot\mathfrak{N} = \{0\}$. Lemma 1.2.1 now implies that $C(\mathfrak{P}_1) \leqq C(\mathfrak{P}) \leqq \overline{C}(\Lambda_2)$. But $\mathfrak{P}_1$ was any closed linear manifold in Λ_1, hence $\overline{C}(\Lambda_1) \leqq \overline{C}(\Lambda_2)$.

To show the second statement, we take $\mathfrak{M} \subseteq \Lambda_0 \oplus \Lambda_1$ as the set of elements $\{0, f\}$ of $\Lambda_0 \oplus \Lambda_1$, E as the transformation of $\Lambda_0 \oplus \Lambda_1$, such that $E\{f, g\} = \{0, g\}$. One readily sees that $\mathfrak{M}$ is equivalent to Λ_1 and that E projects $\Lambda_0 \oplus \Lambda_1$ on $\mathfrak{M}$ and $|E| = 1$. We may now apply the first part of this lemma to obtain the desired result.

LEMMA 1.2.3. *If $\Lambda \cong \sum_{\alpha=1}^{\infty} \oplus \Lambda_\alpha$ and k is* $\limsup_{\alpha\to\infty} \overline{C}(\Lambda_\alpha)$, *then there exists a manifold $\mathfrak{P} \subseteq \Lambda$, such that $C(\mathfrak{P}) \geqq k$.*

It follows from the definition of k, that if ϵ is >0, then there exists an infinite number of the α's for which $r(\overline{C}(\Lambda_\alpha), k) \geqq -\epsilon$. Thus we can find a sequence of integers $\{\alpha_i\}$ such that $\alpha_i < \alpha_{i+1}$, for which $r(\overline{C}(\Lambda_{\alpha_i}), k) \geqq -2^{-i-1}$

Now since $r(\overline{C}(\Lambda_{\alpha_i}), k) \geqq -2^{-i-1}$, we can find a $\mathfrak{P}_{\alpha_i}$, in Λ_{α_i}, such that $r(C(\mathfrak{P}_{\alpha_i}), k) > -2^{-i}$. Let $\mathfrak{P}$ be the closed linear manifold consisting of those elements $\{f_1, f_2, f_3, \cdots\} \epsilon \Lambda$, such that $f_\beta = 0$ if β is not $\epsilon \{\alpha_i\}$ and $f_{\alpha_i} \epsilon \mathfrak{P}_{\alpha_i}$. As we saw in the proof of Lemma 1.2.2, Λ_{α_i} and Λ are as Λ_1 and Λ_2 in Lemma 1.2.1

and it is easily seen that $\mathfrak{P}$ satisfies the conditions given in Lemma 1.2.1 also. Thus Lemma 1.2.1 now implies that $C(\mathfrak{P})$ is $\geqq C(\mathfrak{P}_{\alpha_i})$. Hence $r(C(\mathfrak{P}), k) \geqq -2^{-i}$ for every i. This implies that $r(C(\mathfrak{P}), k) \geqq 0$, $C(\mathfrak{P}) \geqq k$.

1.3. We now prove the following lemma.

LEMMA 1.3.1. $\overline{C}(L_p) \geqq \overline{C}(l_{p,\infty})$.

In (B), Theorem 9, Chapter XII, p. 206, it is shown that the manifold $\mathfrak{M} \subseteq L_p$, determined by the functions y_i is equivalent to l_p when

$$y_i(t) = 2^{i/p} \text{ for } 1/2^i \leqq t \leqq 1/2^{i-1}, \quad y_i(t) = 0, \quad \text{otherwise}.$$

Now for any $z(t) \in L_p$, let

$$E(z(t)) = \sum_{i=1}^{\infty} \int_0^1 z(s) y_i^{p-1}(s) ds \cdot y_i(t).$$

Then by a direct calculation one can verify that $|E| = 1$ and that if $z \in \mathfrak{M}$ (i.e., if $z = \sum_i \alpha_i y_i$, $\sum |\alpha_i|^p < \infty$), then $Ez = z$. Hence E projects L_p on $\mathfrak{M}$ and we may apply Lemma 1.2.2 so that it yields $\overline{C}(L_p) \geqq \overline{C}(l_{p,\infty})$.

LEMMA 1.3.2. *There exists a linear manifold* $\mathfrak{M} \subseteq L_p$, *such that* $C(\mathfrak{M}) = \overline{C}(L_p)$.

This follows from Lemma 1.1.2, (b) (with $n = \infty$) and Lemma 1.2.3 for k is in this case $\overline{C}(L_p)$.

LEMMA 1.3.3. *There exists a linear manifold* $\mathfrak{M} \subseteq l_{p,\infty}$, *such that* $C(\mathfrak{M}) = \overline{C}(l_{p,\infty})$.

In Lemma 1.1.2, (a), let $n_\alpha = \infty$ for every α. Then apply Lemma 1.2.3.

LEMMA 1.3.4. $\overline{C}(l_{p,n}) \geqq \overline{C}(l_{p,m})$ *if* $n \geqq m$.

This follows from Lemma 1.1.2, (a) and Lemma 1.2.2.

THEOREM I. $C(\mathfrak{M})$ *and* $\overline{C}(\Lambda)$ *are to be as in* §1.2. *There exists an* $\mathfrak{M}$ *in* L_p, *and an* $\mathfrak{N}$ *in* l_p, *such that* $C(\mathfrak{M}) = \overline{C}(L_p)$ *and* $C(\mathfrak{N}) = \overline{C}(l_p)$. *Furthermore*

$$1 = \overline{C}(l_{p,1}) \leqq \overline{C}(l_{p,2}) \leqq \cdots \leqq \overline{C}(l_p) \leqq \overline{C}(L_p).$$

The lemmas of this section imply this theorem.

Now if we are able to show that $\lim_{n\to\infty} \overline{C}(l_{p,n}) = \infty$, it follows from this theorem that $\overline{C}(L_p) = \overline{C}(l_p) = \infty$ and then in each of them we have a manifold $\mathfrak{M}$ for which $C(\mathfrak{M}) = \infty$. Hence from the definition of $C(\mathfrak{M})$, we can answer problems (a) and (b) negatively. The next two chapters of this paper contain the proof of the fact that $\lim_{n\to\infty} \overline{C}(l_{p,n}) = \infty$.

CHAPTER 2. $\mathfrak{M}$ IN SITUATION A

2.1. Let $f = \{a_1, \cdots, a_n\}$ be an n-dimensional vector, which may be regarded as $\in l_{p,n}$. We define for $k > 0$

$$\{f\}^k = \{|a_1|^k \operatorname{sign} a_1, \cdots, |a_n|^k \operatorname{sign} a_n\}$$

$$[f]^k = \{|a_1|^k, \cdots, |a_n|^k\},$$

which may be regarded as elements of $l_{p/k,n}$. If $g=\{b_1, \cdots, b_n\}$, we define $(f, g)=\sum_{i=1}^n a_i b_i$. The linearity and homogeneity of this expression will be used without comment.

The following two lemmas can be easily shown.

LEMMA 2.1.1. *If* $p>1$,

$$\frac{d}{dt}\|f+tg\|^p\Big]_{t=0} = p(\{f\}^{p-1}, g).$$

LEMMA 2.1.2. *If* $p>2$,

$$\frac{d^2}{dt^2}\|f+tg\|^p = p(p-1)([f+tg]^{p-2}, [g]^2).$$

We now prove

LEMMA 2.1.3. *If* $p>2$, *and* $(\{f\}^{p-1}, g)\geqq 0$, *then* $\|f+g\|^p \geqq \|f\|^p$.

By Lemma 2.1.2, $H(t)=\|f+tg\|^p$ is convex in t and hence increasing for $t\geqq 0$ since $dH/dt|_{t=0}\geqq 0$ by Lemma 2.1.1.

2.2. If $\mathfrak{M}$ is a linear manifold in $l_{p,n}$, let $\mathfrak{M}^\perp$ consist of those elements $g\epsilon l_{p',n}$, $1/p+1/p'=1$, for which $(f, g)=0$ for all $f\epsilon\mathfrak{M}$. If $\mathfrak{M}$ is k-dimensional, it is well known that $\mathfrak{M}^\perp$ is $(n-k)$-dimensional and also that $(\mathfrak{M}^\perp)^\perp=\mathfrak{M}$. The following lemma is of a standard type in the theory of linear manifolds of a finite number of dimensions and the proof of it may be omitted.

LEMMA 2.2.1. *Let* $\mathfrak{M}$ *be a* k*-dimensional linear manifold in* $l_{p,n}$. *Let* $\phi_1, \cdots, \phi_k$ *be* k *linearly independent elements of* $\mathfrak{M}$. *If* E *is a projection of* $l_{p,n}$ *on* $\mathfrak{M}$, *there exist* k *elements* $\psi_1, \cdots, \psi_k$, *of* $l_{p',n}$ *such that for every* $f\epsilon l_{p,n}$,

$$(\alpha) \qquad Ef = \sum_{i=1}^{k} (\psi_i, f)\phi_i$$

and $(\psi_i, \phi_j)=\delta_{i,j}$. *If* ψ_i*'s with this last property are given, the* E *defined by* (α) *is a projection. If* E' *is any other projection of* $l_{p,n}$ *on* $\mathfrak{M}$, *then* $\psi_i'=\psi_i+g_i$, $i=1, \cdots, k$, *where* g_i *is* $\epsilon\mathfrak{M}^\perp$.

2.3. If E is a linear transformation in $l_{p,n}$, we denote by E^* (the adjoint of E) the transformation in $l_{p',n}$, such that if g and $g^*\epsilon l_{p',n}$, are related so that for every $f\epsilon l_{p,n}$, $(Ef, g)=(f, g^*)$, then $E^*g=g^*$. It is well known that E^* is linear, $|E|=|E^*|$ and $(FE)^*=E^*F^*$.

LEMMA 2.3.1. *If* $\mathfrak{M}$ *and* E *are as in Lemma* 2.2.1, *then*

$$E^*g = \sum_{i=1}^{k} (\phi_i, g)\psi_i$$

for all $g \epsilon l_{p',n}$.

For every $f \epsilon l_{p,n}$, we have

$$(Ef, g) = \left(\sum_{i=1}^{k} (\psi_i, f)\phi_i, g \right) = \sum_{i=1}^{k} (\psi_i, f)(\phi_i, g) = \left(f, \sum_{i=1}^{k} (\phi_i, g)\psi_i \right).$$

LEMMA 2.3.2. *If* $\mathfrak{M}$ *and* E *are as in Lemma* 2.2.1, *then* $1-E^*$ *is a projection on* $\mathfrak{M}^{\perp}$.

The range of $1-E^*$ is $\mathfrak{M}^{\perp}$. For if $f=(1-E^*)g$ and h is $\epsilon\mathfrak{M}$, then $Eh=h$ and

$$(h, f) = (h, (1 - E^*)g) = (h, g) - (h, E^*g) = (Eh, g) - (h, E^*g) = 0.$$

Hence f is $\epsilon\mathfrak{M}^{\perp}$ or the range of $1-E^*$ is included in $\mathfrak{M}^{\perp}$. Furthermore by Lemma 2.3.1, if f is $\epsilon\mathfrak{M}^{\perp}$,

$$E^*f = \sum_{i=1}^{k} (\phi_i, f)\psi_i = 0$$

and $(1-E^*)f=f$. This with our previous result shows that the range $1-E^*$ is exactly $\mathfrak{M}^{\perp}$ and $(1-E^*)^2=1-E^*$.

LEMMA 2.3.3. $\overline{C}(l_{p',n}) \leqq \overline{C}(l_{p,n})+1$.

Let $\mathfrak{M}'$ be any linear manifold of $l_{p',n}$. Let $\mathfrak{M}=\mathfrak{M}'^{\perp}$. Then if ϵ is >0, there exists a projection E of $l_{p,n}$ on $\mathfrak{M}$ with $|E| \leqq C(\mathfrak{M})+\epsilon \leqq \overline{C}(l_{p,n})+\epsilon$. By Lemma 2.3.2, $1-E^*$ is a projection on $\mathfrak{M}^{\perp}=(\mathfrak{M}'^{\perp})^{\perp}=\mathfrak{M}'$. Thus $C(\mathfrak{M}') \leqq |1-E^*| \leqq 1+|E^*| = 1+|E| \leqq 1+\overline{C}(l_{p,n})+\epsilon$, which implies our lemma.

Of course p and p' are interchangeable and so we see that the answer to our question is the same for both p and p'. Thus we may confine ourselves to the case $p<2$. This is not an essential step in our proof but merely a convenient one. We suppose from now on that p is <2.

2.4. We say that Situation A holds in a k-dimensional manifold $\mathfrak{M}$ of $l_{p,n}$, if

(a) we have k linearly independent elements, $\phi_1, \cdots, \phi_k$, $\epsilon\mathfrak{M}$, and k elements of $l_{p',n}$, $\psi_1, \cdots, \psi_k$ such that $(\phi_i, \psi_j)=\delta_{i,j}$ (the transformation E given by the equation $Ef=\sum_{i=1}^{k}(\psi_i, f)\phi_i$ is a projection of $l_{p,n}$ on $\mathfrak{M}$);

(b) we have r elements $h_1, \cdots, h_r$ of $\mathfrak{M}$, with $\|h_i\|=1$;

(c) there exists a constant $C>1$, such that $\|E^*\{h_i\}^{p-1}\|=C$ for every i;

(d) there exist r constants $c_1, \cdots, c_r$, $c_i>0$, such that for every $f\epsilon\mathfrak{M}$ and $g\epsilon\mathfrak{M}^{\perp}$

$$\sum_{i=1}^{r} c_i(\{h_i\}^{p-1}, f)(\{E^*\{h_i\}^{p-1}\}^{p'-1}, g) = 0.$$

LEMMA 2.4.1. *If* $\mathfrak{M}$ *is in Situation* A, *then* $C(\mathfrak{M}) \geqq C$ (cf. (c) above).

Since $|E| = |E^*|$, we must show that for every projection E' of $l_{p,n}$ on $\mathfrak{M}$, $|E'^*| \geqq C$. Since $\|h_i\| = 1$ and hence $\|\{h_i\}^{p-1}\| = 1$, it will be sufficient to show that $\|E'^*\{h_i\}^{p-1}\| \geqq C$ for at least one i.

Now

$$E'^*\{h_i\}^{p-1} = \sum_{j=1}^{k} (\{h_i\}^{p-1}, \phi_j)\psi_j',$$

where $\psi_j' = \psi_j + g_j$, where g_j is an element of $\mathfrak{M}^\perp$ (Lemmas 2.2.1 and 2.3.1). Let E_t be the projection of $l_{p,n}$ on $\mathfrak{M}$ given by

$$E_t f = \sum_{i=1}^{k} (\psi_i + t g_i, f)\phi_i.$$

By Lemma 2.3.1,

$$E_t^* g = \sum_{i=1}^{k} (\phi_i, g)(\psi_i + t g_i)$$

and

$$\begin{aligned} E_t^*\{h_i\}^{p-1} &= \sum_{j=1}^{k} (\phi_j, \{h_i\}^{p-1})(\psi_j + t g_j) \\ &= \sum_{j=1}^{k} (\phi_j, \{h_i\}^{p-1})\psi_j + t\sum_{j=1}^{k} (\phi_j, \{h_i\}^{p-1})g_j \\ &= E^*\{h_i\}^{p-1} + t\sum_{j=1}^{k} (\phi_j, \{h_i\}^{p-1})g_j. \end{aligned}$$

Now by Lemma 2.1.1

$$\begin{aligned} \frac{d}{dt}\|E_t^*\{h_i\}^{p-1}\|_{t=0}^{p'} &= \left(\{E^*\{h_i\}^{p-1}\}^{p'-1}, \sum_{j=1}^{k} (\phi_j, \{h_i\}^{p-1})g_j\right) \\ &= \sum_{j=1}^{k} (\{E^*\{h_i\}^{p-1}\}^{p'-1}, g_j)(\phi_j, \{h_i\}^{p-1}). \end{aligned}$$

Since $g_j \epsilon \mathfrak{M}^\perp$, $\phi_j \epsilon \mathfrak{M}$, (d) implies that

$$\sum_{i=1}^{r} c_i \frac{d}{dt}\|E_t^*\{h_i\}^{p-1}\|^{p'}]_{t=0} = \sum_{i=1}^{r} c_i \sum_{j=1}^{k} (\{E^*\{h_i\}^{p-1}\}^{p'-1}, g_j)(\phi_j, \{h_i\}^{p-1}) = 0.$$

Since $c_i > 0$, for $i = 1, \cdots, r$ this implies that there must be an i' such that $d\|E_t^*\{h_{i'}\}^{p-1}\|^{p'}/dt]_{t=0}$ is $\geqq 0$. Hence by the above $d\|E^*\{h_{i'}\}^{p-1} + tg'_{i'}\|^{p'}/dt]_{t=0}$ is $\geqq 0$, when $g'_{i'} = \sum_{i=1}^{k}(\phi_i, \{h_{i'}\}^{p-1})g_i$. Lemma 2.1.3 now yields

$$\|E^*\{h_{i'}\}^{p-1} + g'_{i'}\|^{p'} \geqq \|E^*\{h_{i'}\}^{p-1}\|^{p'}$$

since $p' > 2$. But

$$E'^*\{h_{i'}\}^{p-1} = E_1^*\{h_{i'}\}^{p-1} = E^*\{h_{i'}\}^{p-1} + g'_{i'}$$

and $\|E^*\{h_{i'}\}^{p-1}\|^{p'} = C$. Substituting these values on both sides of our inequality, we get $\|E'^*\{h_{i'}\}^{p-1}\|^{p'} \geqq C^{p'}$ or $\|E'^*\{h_{i'}\}^{p-1}\| \geqq C$. As we remarked at the beginning of the proof this is sufficient.

Chapter 3. The product of $l_{p,n}$ and $l_{p,m}$

3.1. We define $l_{p,n} \otimes l_{p,m}$ as $l_{p,nm}$. If $f = \{a_1, \cdots, a_n\} \in l_{p,n}$, and $g = \{b_1, \cdots, b_m\} \in l_{p,m}$, we define $f \otimes g$ as $\{a_1b_1, a_1b_2, \cdots, a_1b_m, a_2b_1, a_2b_2, \cdots, a_2b_m, \cdots, a_nb_1, a_nb_2, \cdots, a_nb_m\}$ or if $f \otimes g = \{c_1, \cdots, c_{nm}\}$, then $c_{(s-1)m+t} = a_sb_t$. The proofs of the following Lemmas 3.1.1–3.1.4 do not present any difficulty and may be left to the reader.

Lemma 3.1.1. (i) $\|f \otimes g\| = \|f\| \cdot \|g\|$,
(ii) $(\phi_1 \otimes \phi_2, f \otimes g) = (\phi_1, f)(\phi_2, g)$,
(iii) $\{f \otimes g\}^k = \{f\}^k \otimes \{g\}^k$,
(iv) $\alpha(f^{(1)} \otimes g) + \beta(f^{(2)} \otimes g) = (\alpha f^{(1)} + \beta f^{(2)}) \otimes g$.

Lemma 3.1.2. *Let $f_r = \{a_{1,r}, \cdots, a_{n,r}\}$, $r = 1, \cdots, k$, $k \leqq n$, be k linearly independent elements of $l_{p,n}$, and $g = \{b_{1,r}, \cdots, b_{m,r}\}$ be k elements of $l_{p,m}$, such that $\sum_{r=1}^{k} f_r \otimes g_r = 0$. Then $g_r = 0$, $r = 1, \cdots, k$.*

Lemma 3.1.3. *Let f_r, $r = 1, \cdots, k$, $k \leqq n$, be k linearly independent elements of $l_{p,n}$ and for every $r = 1, \cdots, k$ let $g_{r,s}$, $s = 1, \cdots, k_r$, $k_r \leqq m$, be k_r linearly independent elements of $l_{p,m}$. Then the set of elements $f_r \otimes g_{r,s}$, $r = 1, \cdots, k$, $s = 1, \cdots, k$ are linearly independent.*

Lemma 3.1.4. *Let $f_1, \cdots, f_n$ be n linearly independent elements of $l_{p,n}$, $g_1, \cdots, g_m$, m linearly independent elements $l_{p,m}$. Then the set of elements $f_i \otimes g_j$, $i = 1, \cdots, n$, $j = 1, \cdots, m$, determine $l_{p,mn}$.*

3.2. Let $\mathfrak{M}^{(1)} \subseteq l_{p,n}$ and $\mathfrak{M}^{(2)} \subseteq l_{p,m}$ be linear manifolds. We define $\mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)}$ as the linear manifold in $l_{p,mn}$ determined by the elements $f \otimes g$, $f \in \mathfrak{M}^{(1)}$, $g \in \mathfrak{M}^{(2)}$.

Lemma 3.2.1. *If $\phi_1^{(1)}, \cdots, \phi_{k^{(1)}}^{(1)}$ is a set of linearly independent elements which determine $\mathfrak{M}^{(1)}$ and $\phi_1^{(2)}, \cdots, \phi_{k^{(2)}}^{(2)}$ is a set of linearly independent elements which determine $\mathfrak{M}^{(2)}$, then $\phi_i^{(1)} \otimes \phi_j^{(2)}$, $i = 1, \cdots, k^{(1)}$, $j = 1, \cdots, k^{(2)}$, determine the manifold $\mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)}$ which is $k^{(1)}k^{(2)}$-dimensional.*

The proof is easily derived from Lemma 3.1.3.

LEMMA 3.2.2. *Let $\mathfrak{M}^{(1)}$ and $\mathfrak{M}^{(2)}$ be as above. Then $(\mathfrak{M}^{(1)}\otimes\mathfrak{M}^{(2)})^\perp$ is determined by the elements of the form $f\otimes g$, where either $f\epsilon\mathfrak{M}^{(1)\perp}$ or $g\epsilon\mathfrak{M}^{(2)\perp}$.*

We first show that if $f\otimes g$ is such that either $f\epsilon\mathfrak{M}^{(1)\perp}$ or $g\epsilon\mathfrak{M}^{(2)\perp}$, then $f\otimes g\epsilon(\mathfrak{M}^{(1)}\otimes\mathfrak{M}^{(2)})^\perp$. Indeed by the definition of $\mathfrak{M}^{(1)}\otimes\mathfrak{M}^{(2)}$ and the linearity of the operation (,), if $(\phi^{(1)}\otimes\phi^{(2)}, f\otimes g)=0$ for all $\phi^{(1)}\epsilon\mathfrak{M}^{(2)}$ and $\phi^{(2)}\epsilon\mathfrak{M}^{(2)}$, then $f\otimes g\epsilon(\mathfrak{M}^{(1)}\otimes\mathfrak{M}^{(2)})^\perp$. But Lemma 3.1.1, (ii) implies that $(\phi^{(1)}\otimes\phi^{(2)}, f\otimes g)=(\phi^{(1)}, f)(\phi^{(2)}, g)$. Since either $(\phi^{(1)}, f)=0$ or $(\phi^{(2)}, g)=0$, we obtain that $f\otimes g\epsilon(\mathfrak{M}^{(1)}\otimes\mathfrak{M}^{(2)})^\perp$.

Now let $\tilde{\phi}_1^{(1)}, \cdots, \tilde{\phi}_{n-k^{(1)}}^{(1)}$ and $\tilde{\phi}_1^{(2)}, \cdots, \tilde{\phi}_{n-k^{(2)}}^{(2)}$ be sets of $n-k^{(1)}$ and $m-k^{(2)}$ linearly independent elements of $\mathfrak{M}^{(1)\perp}$ and $\mathfrak{M}^{(2)\perp}$ respectively. Let $\psi_1^{(1)}, \cdots, \psi_{k^{(1)}}^{(1)}$ $(\psi_1^{(2)}, \cdots, \psi_{k^{(2)}}^{(2)})$ be elements of $l_{p',n}$ $(l_{p',m})$ such that $\tilde{\phi}_i^{(1)}$'s and $\psi_j^{(1)}$'s together determine $l_{p',n}$ ($\tilde{\phi}_i^{(2)}$'s and $\psi_j^{(2)}$'s determine $l_{p',m}$). In Lemma 3.1.3, let $f_r=\psi_r^{(1)}$ for $r=1, \cdots, k^{(1)}$, $f_{k^{(1)}+t}=\tilde{\phi}_t^{(1)}$ for $t=1, \cdots, n-k^{(1)}$; $k=n$. For $r=1, \cdots, k^{(1)}$, let $k_r=m-k^{(2)}$, $g_{r,s}=\tilde{\phi}_s^{(2)}$, $s=1, \cdots, m-k^{(2)}$, and for $r=k+1, \cdots, n$; $k=m$, $g_{r,s}=\psi_s^{(2)}$, $s=1, \cdots, k^{(2)}$, $g_{r,k^{(2)}+t}=\phi_t^{(2)}$, $t=1, \cdots, m-k^{(2)}$. Thus the $f_r\otimes g_{r,s}$ are linearly independent and such that either $f_r\epsilon\mathfrak{M}^{(1)\perp}$, or $g_{r,s}\epsilon\mathfrak{M}^{(2)\perp}$. Since there are $k^{(1)}(m-k^{(2)})+(n-k^{(1)})m=mn-k^{(1)}k^{(2)}$ of them and the dimensionality of $\mathfrak{M}^{(1)}\otimes\mathfrak{M}^{(2)}$ is $k^{(1)}k^{(2)}$ by Lemma 3.2.1, the $f_r\otimes g_{r,s}$ determine $(\mathfrak{M}^{(1)}\otimes\mathfrak{M}^{(2)})^\perp$.

3.3. Let $T^{(1)}$ be a linear transformation in $l_{p,n}$ and $T^{(2)}$ a linear transformation in $l_{p,m}$. Let $\phi_1^{(1)}, \cdots, \phi_n^{(1)}$ be n linearly independent elements of $l_{p,n}$ and $\phi_1^{(2)}, \cdots, \phi_m^{(2)}$, m linearly independent elements of $l_{p,m}$. Then the elements $\phi_i^{(1)}\otimes\phi_j^{(2)}$, $i=1, \cdots, n, j=1, \cdots, m$ determine $l_{p,mn}$ by Lemma 3.1.4 and are linearly independent. Hence given any set of mn elements $f_{i,j}\epsilon l_{p,mn}$, there exists a unique linear transformation T' such that

$$T'(\phi_i^{(1)} \otimes \phi_j^{(2)}) = f_{i,j} \qquad (i = 1, \cdots, n, j = 1, \cdots, m).$$

Now let $f_{i,j}=T^{(1)}\phi_i^{(1)}\otimes T^{(2)}\phi_j^{(2)}$ and denote the corresponding T' by $T^{(1)}\otimes T^{(2)}$. Apparently this definition of $T^{(1)}\otimes T^{(2)}$ depends on the choice of the $\phi_i^{(1)}$'s and $\phi_j^{(2)}$'s, but this is not the case as is shown by the following

LEMMA 3.3.1. *If $f\epsilon l_{p,n}$, $g\epsilon l_{p,m}$, then $T^{(1)}\otimes T^{(2)}f\otimes g=T^{(1)}f\otimes T^{(2)}g$. Thus $T^{(1)}\otimes T^{(2)}$ does not depend on the choice of the $\{\phi_i^{(1)}\}$ or the $\{\phi_j^{(2)}\}$.*

The proof follows immediately from the definition of $T^{(1)}\otimes T^{(2)}$ and Lemma 3.1.1, (iv).

We also have

LEMMA 3.3.2. *A linear transformation T' of $l_{p,mn}$, equals $T^{(1)}\otimes T^{(2)}$ if and only if $T'f\otimes g=T^{(1)}f\otimes T^{(2)}g$ for every $f\epsilon l_{p,n}$, and $g\epsilon l_{p,m}$.*

The sufficiency of the condition follows from the definition. Lemma 3.3.1 implies its necessity.

LEMMA 3.3.3. $(T^{(1)}\otimes T^{(2)})^*=(T^{(1)})^*\otimes(T^{(2)})^*$

By Lemma 3.3.2, it suffices to show that if $f\epsilon l_{p',n}$, $g\epsilon l_{p',m}$, then $(T^{(1)}\otimes T^{(2)})^*f\otimes g=T^{(1)*}f\otimes T^{(2)*}g$. Now if $h\epsilon l_{p,mn}$, it follows from Lemma 3.1.4, that $h=\sum_{i=1}^n\sum_{j=1}^m a_{i,j}\phi_i^{(1)}\otimes\phi_j^{(2)}$ and by the definition of $T^{(1)}\otimes T^{(2)}$, $T^{(1)}\otimes T^{(2)}h=\sum_{i=1}^n\sum_{j=1}^m a_{i,j}T^{(1)}\phi_i^{(1)}\otimes T^{(2)}\phi_j^{(2)}$.

By the definition of T^* (cf. §2.3) and Lemma 3.1.1, (ii), we have

$$\begin{aligned}(T^{(1)}\otimes T^{(2)}h, f\otimes g) &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}(T^{(1)}\phi_i^{(1)}\otimes T^{(2)}\phi_j^{(2)}, f\otimes g)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}(T^{(1)}\phi_i^{(1)}, f)(T^{(2)}\phi_j^{(2)}, g)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}(\phi_i^{(1)}, T^{(1)*}f)(\phi_j^{(2)}, T^{(2)*}g)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}(\phi_i^{(1)}\otimes\phi_j^{(2)}, T^{(1)*}f\otimes T^{(2)*}g)\\ &= \left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j}\phi_i^{(1)}\otimes\phi_j^{(2)}, T^{(1)*}f\otimes T^{(2)*}g\right).\\ &= (h, T^{(1)*}f\otimes T^{(2)*}g).\end{aligned}$$

Or for every h of $l_{p,mn}$,

$$(T^{(1)}\otimes T^{(2)}h, f\otimes g) = (h, T^{(1)*}f\otimes T^{(2)*}g).$$

The definition of T^*, then implies

$$(T^{(1)}\otimes T^{(2)})^*f\otimes g = T^{(1)*}f\otimes T^{(2)*}g$$

which is the desired result.

3.4. We have the following lemma.

LEMMA 3.4.1. (i) *If $E^{(1)}$ is a projection on $\mathfrak{M}^{(1)}\subseteq l_{p,n}$, and $E^{(2)}$ a projection on $\mathfrak{M}^{(2)}\subseteq l_{p,m}$, then $E^{(1)}\otimes E^{(2)}$ is a projection of $l_{p,mn}$ on $\mathfrak{M}^{(1)}\otimes\mathfrak{M}^{(2)}$.*

(ii) *Let $\phi_i^{(1)}$ and $\psi_i^{(1)}$ ($\phi_i^{(2)}$ and $\psi_i^{(2)}$) be in the same relation to $E^{(1)}$ ($E^{(2)}$) as ϕ_i and ψ_i are to E in Lemma 2.2.1, i.e.,*

$$E^{(1)}f = \sum_{i=1}^{k^{(1)}}(\psi_i^{(1)}, f)\phi_i^{(1)}; \qquad E^{(2)}f = \sum_{j=1}^{k^{(2)}}(\psi_j^{(2)}, f)\phi_j^{(2)}.$$

If E is the transformation on $l_{p,mn}$ defined by the equation

$$Eh = \sum_{i=1}^{k^{(1)}} \sum_{j=1}^{k^{(2)}} (\psi_i^{(1)} \otimes \psi_j^{(2)}, h)\phi_i^{(1)} \otimes \phi_j^{(2)},$$

then E is a projection of $l_{p,mn}$ on $\mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)}$ and $E = E^{(1)} \otimes E^{(2)}$.

(iii) $E^* = E^{(1)*}E^{(2)*}$.

Since (ii) implies (i) and (ii) and Lemma 3.3.3 implies (iii), we need only prove (ii).

We have $(\psi_i^{(1)} \otimes \psi_j^{(2)}, \phi_l^{(1)} \otimes \phi_k^{(2)}) = (\psi_i^{(1)}, \phi_l^{(1)})(\psi_j^{(2)}, \phi_k^{(2)}) = \delta_{i,l}\delta_{j,k}$, by Lemma 3.1.1 and Lemma 2.2.1. By Lemma 3.2.1 the $\phi_l^{(1)} \otimes \phi_k^{(2)}$ determine $\mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)}$. Lemma 2.2.1 now implies that E is a projection on $\mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)}$.

It remains to show that $E = E^{(1)} \otimes E^{(2)}$. If $f \epsilon l_{p,n}$, $g \epsilon l_{p,m}$, then by Lemma 3.1.1, (ii) and (iv), and Lemma 2.2.1,

$$\begin{aligned} Ef \otimes g &= \sum_{i=1}^{k^{(1)}} \sum_{j=1}^{k^{(2)}} (\psi_i^{(1)} \otimes \psi_j^{(2)}, f \otimes g)\phi_i^{(1)} \otimes \phi_j^{(2)} \\ &= \sum_{i=1}^{k^{(1)}} \sum_{j=1}^{k^{(2)}} (\psi_i^{(1)}, f)(\psi_j^{(2)}, g)\phi_i^{(1)} \otimes \phi_j^{(2)} \\ &= \left(\sum_{i=1}^{k^{(1)}} (\psi_i^{(1)}, f)\phi_i^{(1)} \right) \otimes \left(\sum_{j=1}^{k^{(2)}} (\psi_j^{(2)}, g)\phi_j^{(2)} \right) = E^{(1)}f \otimes E^{(2)}g. \end{aligned}$$

Lemma 3.3.2 now implies that $E = E^{(1)} \otimes E^{(2)}$.

3.5. Next we prove

LEMMA 3.5.1. *If $\mathfrak{M}^{(1)}$ in $l_{p,n}$ and $\mathfrak{M}^{(2)}$ in $l_{p,m}$ are in Situation* A (cf. §2.4), *then $\mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)}$ is in Situation* A *with*

(a) $\phi_{(s-1)k^{(1)}+t} = \phi_t^{(1)} \otimes \phi_j^{(2)}, \quad t = 1, \cdots, k^{(1)}, \quad s = 1, \cdots, k^{(2)},$

$\psi_{(s-1)k^{(1)}+t} = \psi_t^{(1)} \otimes \psi_s^{(2)}, \quad t = 1, \cdots, k^{(1)}, \quad s = 1, \cdots, k^{(2)};$

(b) $h_{(s-1)r^{(1)}+t} = h_t^{(1)} \otimes h_s^{(2)}, \quad t = 1, \cdots, r^{(1)}, \quad s = 1, \cdots, r^{(2)};$

(c) $C = C^{(1)}C^{(2)};$

(d) $c_{(s-1)r^{(1)}+t} = c_t^{(1)} c_s^{(2)}, \qquad t = 1, \cdots, r^{(1)}; \quad s = 1, \cdots, r^{(2)}.$

That the $\phi_t^{(1)} \otimes \phi_s^{(2)}$, $t = 1, \cdots, k^{(1)}$, $s = 1, \cdots, k^{(2)}$ are linearly independent and determine $\mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)}$ has been shown in Lemma 3.1.3 and Lemma 3.2.1. The remaining statements of (a) were shown in the proof of Lemma 3.4.1.

To prove (b) we have $h_t^{(1)} \otimes h_s^{(2)} \epsilon \mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)}$ by definition (cf., §3.2). Also by Lemma 3.1.1 $\|h_t^{(1)} \otimes h_s^{(2)}\| = \|h_t^{(1)}\| \cdot \|h_s^{(2)}\| = 1$.

Now consider (c). If $i = (s-1)r^{(1)} + t$, then by Lemma 3.1.1, (ii); Lemma 3.4.1, (iii); Lemma 3.3.2; Lemma 3.1.1, (i); and §2.4, (c),

$$\begin{aligned}\|E^*\{h_i\}^{p-1}\| &= \|E^*\{h_t^{(1)} \otimes h_s^{(2)}\}^{p-1}\| = \|E^*\{h_t^{(1)}\}^{p-1} \otimes \{h_s^{(2)}\}^{p-1}\| \\ &= \|E^{(1)*} \otimes E^{(2)*}\{h_t^{(1)}\}^{p-1} \otimes \{h_s^{(2)}\}^{p-1}\| \\ &= \|E^{(1)*}\{h_t^{(1)}\}^{p-1} \otimes E^{(2)*}\{h_s^{(2)}\}^{p-1}\| \\ &= \|E^{(1)*}\{h_t^{(1)}\}^{p-1}\| \cdot \|E^{(2)*}\{h_t^{(2)}\}^{p-2}\|.\end{aligned}$$

We now prove (d). If $h \epsilon (\mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)})^{\perp}$, then by Lemma 3.2.2, h is a linear combination of elements in the form $f^{(1)} \otimes g^{(2)}$, where either $f^{(1)}$ is in $\mathfrak{M}^{(1)\perp}$ or $g^{(2)}$ is in $\mathfrak{M}^{(2)\perp}$. Hence since

$$\sum_{i=1}^{r^{(1)}r^{(2)}} c_i(\{h_i\}^{p-1}, f)(h, \{E^*\{h_i\}^{p-1}\}^{p'-1}),$$

$f \epsilon \mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)}$, $h \epsilon (\mathfrak{M}^{(1)} \otimes \mathfrak{M}^{(2)})^{\perp}$, is linear in h, it is enough to show (d) for h in the form $f^{(1)} \otimes g^{(2)}$, where either $f^{(1)}$ is in $\mathfrak{M}^{(1)\perp}$ or $g^{(2)}$ is in $\mathfrak{M}^{(2)\perp}$. It is also linear in f; hence by §3.2 it suffices to show (d) for f in the form $\phi^{(1)} \otimes \phi^{(2)}$, $\phi_1 \epsilon \mathfrak{M}^{(1)}$, $\phi_2 \epsilon \mathfrak{M}^{(2)}$.

Now it was shown in the proof of (c) above that $E^*\{h_i\}^{p-1} = E^{(1)*}\{h_t^{(1)}\}^{p-1} \otimes E^{(2)*}\{h_s^{(2)}\}^{p-1}$. Then by Lemma 3.1.1, (iii),

$$\{E^*\{h_i\}^{p-1}\}^{p'-1} = \{E^{(1)*}\{h_t^{(1)}\}^{p-1}\}^{p'-1} \otimes \{E^{(2)*}\{h_s^{(2)}\}^{p-1}\}^{p'-1}$$

and $\{h_i\}^{p-1} = \{h_t^{(1)}\}^{p-1} \otimes \{h_s^{(2)}\}^{p-1}$. Hence by Lemma 3.1.1, (ii), we see that

$$\begin{aligned}&\sum_{i=1}^{r^{(1)}r^{(2)}} c_i(\{h_i\}^{p-1}, \phi^{(1)} \otimes \phi^{(2)})(f^{(1)} \otimes g^{(2)}, \{E^*\{h_i\}^{p-1}\}^{p'-1}) \\ &\quad = \left(\sum_{t=1}^{r^{(1)}} c_t^{(1)}(\{h_t^{(1)}\}^{p-1}, \phi^{(1)})(f^{(1)}, \{E^{(1)*}\{h_t^{(1)}\}^{p-1}\}^{p'-1}\right) \\ &\quad \times \left(\sum_{s=1}^{r^{(2)}} c_s^{(2)}(\{h_s^{(2)}\}^{p-1}, \phi^{(2)})(g^{(2)}, \{E^{(2)*}\{h_t^{(2)}\}^{p-1}\}^{p'-1}\right).\end{aligned}$$

Since either $f^{(1)} \epsilon \mathfrak{M}^{(1)\perp}$, or $g^{(2)} \epsilon \mathfrak{M}^{(2)\perp}$, this is zero for $\mathfrak{M}^{(1)}$ and $\mathfrak{M}^{(2)}$ are in Situation A.

3.6. Now it follows from Lemma 2.4.1 and Lemma 3.5.1, that we can show that $\lim_{n\to\infty} (\overline{C}(l_{p,n})) = \infty$ if we can find a $\mathfrak{M}$ in $l_{p,n}$ in Situation A (cf. §2.4, (c)). For let N be any integer >0. Then using Lemma 3.5.1 we can find a manifold $\mathfrak{N}_N$ in l_{p,n^N} in Situation A with $C_{\mathfrak{N}_N} = C_{\mathfrak{M}}^N$. Lemma 2.4.1 now implies that $C(\mathfrak{N}_N) \geqq C_{\mathfrak{M}}^N$ and since

$$\overline{C}(l_{p,n^N}) \geqq C(\mathfrak{N}_N), \quad \lim_{n\to\infty} (\overline{C}(l_{p,n})) = \infty.$$

As we remarked in §1.3, this implies that the answer to both (a) and (b) is negative.

Let $\mathfrak{M}$ be the manifold in $l_{p,3}$, determined by the vectors (1, 1, 0) and (0, 1, 1). Let

$$\phi_1 = (2^{-1/p}, 2^{-1/p}, 0), \qquad \phi_2 = (0, 2^{-1/p}, 2^{-1/p}),$$
$$\psi_1 = (2^{1/p+1}/3,\ 2^{1/p}/3,\ -2^{1/p}/3), \quad \psi_2 = (-2^{1/p}/3,\ 2^{1/p}/3,\ 2^{1/p+1}/3);$$

if $\alpha = 1/(2+2^p)^{1/p}$, $h_1 = (\alpha, -\alpha, -2\alpha)$, $h_2 = (\alpha, 2\alpha, \alpha)$, $h_3 = (2\alpha, \alpha, -\alpha)$. Also

$$C = ((2^{p'-1}+1)/3)^{1/p'}((2^{p-1}+1)/3)^{1/p},$$
$$c_1 = c_2 = c_3 = 1.$$

We show that $\mathfrak{M}$ is in Situation A (§2.4) and thus complete the proof.

We have (a) $(\phi_i, \psi_j) = \delta_{i,j}$.

(b) $h_1 = \alpha 2^{1/p}(\phi_1 - 2\phi_2)$, $h_2 = \alpha 2^{1/p}(\phi_1 + \phi_2)$, $h_3 = \alpha 2^{1/p}(2\phi_1 - \phi_2)$, and thus h_i is $\epsilon\mathfrak{M}$, $i = 1, 2, 3$. We also have $\|h_i\| = 1$, $i = 1, 2, 3$.

Before showing (c) and (d) we make certain calculations. From the definitions in §2.1, we get $\{h_1\}^{p-1} = \alpha^{p-1}(1, -1, -2^{p-1})$, $\{h_2\}^{p-1} = \alpha^{p-1}(1, 2^{p-1}, 1)$, $\{h_3\}^{p-1} = (2^{p-1}, 1, 1)$, $(\{h_1\}^{p-1}, \phi_1) = (\{h_3\}^{p-1}, \phi_2) = 0$, $-(\{h_1\}^{p-1}, \phi_2) = (\{h_2\}^{p-1}, \phi_1) = (\{h_2\}^{p-1}, \phi_2) = (\{h_3\}^{p-1}, \phi_1) = 2^{-1}(1 + 2^{p-1})^{1/p}$.

By Lemma 2.3.1

$$E^*\{h_1\}^{p-1} = (\{h_1\}^{p-1}, \phi_1)\psi_1 + (\{h_1\}^{p-1}, \phi_2)\psi_2 = (\{h_1\}^{p-1}, \phi_2)\psi_2$$
$$= -3^{-1}2^{-1/p'}(2^{p-1}+1)^{1/p}(-1, 1, 2),$$
$$E^*\{h_2\}^{p-1} = (\{h_2\}^{p-1}, \phi_1)\psi_1 + (\{h_2\}^{p-1}, \phi_2)\psi_2 = 2^{-1}(2^{p-1}+1)^{1/p}(\psi_1 + \psi_2)$$
$$= 3^{-1}2^{-1/p'}(2^{p-1}+1)^{1/p}(1, 2, 1).$$

Similarly

$$E^*\{h_3\}^{p-1} = 3^{-1}2^{-1/p'}(2^{p-1}+1)^{1/p}(2, 1, -1).$$

Finally (cf. §2.1)

$$\{E^*\{h_1\}^{p-1}\}^{p'-1} = -3^{-(p'-1)}2^{-(p'-1)/p'}(2^{p-1}+1)^{(p'-1)/p}(-1, 1, 2^{p'-1})$$
$$\equiv -K(-1, 1, 2^{p'-1})$$
$$\{E^*\{h_2\}^{p-1}\}^{p'-1} = K(1, 2^{p'-1}, 1)$$
$$\{E^*\{h_3\}^{p-1}\}^{p'-1} = K(2^{p'-1}, 1, -1).$$

(c) By direct calculation, we obtain $\|E^*\{h_1\}^{p-1}\| = \|E^*\{h_2\}^{p-1}\| = \|E^*\{h_3\}^{p-1}\| = C$ using the above. For $p \neq 2$, C is >1 since by the Hölder inequality,

$$6C = (2^p+2)^{1/p}(2^{p'}+2)^{1/p'} \geqq 2\cdot 2 + 2^{1/p}2^{1/p'} = 6,$$

where the equality sign holds only for $p = 2$.

(d) Now if $f \epsilon \mathfrak{M}^\perp$, then $f = k\tilde{\phi}$, where $\tilde{\phi} = (1, -1, 1)$. Thus

$$(\{E^*\{h_1\}^{p-1}\}^{p'-1}, f) = Kk(2 - 2^{p'-1}),$$
$$(\{E^*\{h_2\}^{p-1}\}^{p'-1}, f) = Kk(2 - 2^{p'-1}),$$
$$(\{E^*\{h_3\}^{p-1}\}^{p'-1}, f) = Kk(2^{p'-1} - 2).$$

We can now verify by a direct calculation that (d) holds.

Conclusion. Our results permit us to conclude that

There exists a manifold $\mathfrak{M}_0$ *in* l_p *and* L_p *such that there exists no biorthogonal set* $\{\phi_i, \psi_i\}$ *where* $\{\phi_i\}$ *is a basis for* $\mathfrak{M}_0$ (cf. (B), Chapter VII, p. 110, §3), *while the expansion*

$$(*) \qquad \sum_{i=1}^{\infty} a_i\phi_i, \qquad a_i = (f, \psi_i),$$

converges for each $f \epsilon l_p$ *or* L_p.

Let us suppose that (*) converges for every f. Let $\mathfrak{M}$ be the manifold determined by the ϕ_i's. The ϕ_i's are a basis for $\mathfrak{M}$ (cf. (B), loc. cit.) for if $f \epsilon \mathfrak{M}$, then

$$f = \sum_{i=1}^{\infty} a_i\phi_i, \qquad a_i = (f, \psi_i)$$

by (B), Chapter VII, Theorem 2, p. 107.

We will show that under these circumstances $C(\mathfrak{M})$ is $<\infty$. For let E be the transformation defined by the equation

$$Ef = \sum_{i=1}^{\infty} (f, \psi_i)\phi_i.$$

Ef is defined for every f since we assume that the series is convergent for every f. The same assumption implies that E is limited for the partial sums are uniformly limited (cf. (B), Chapter VII, Theorem 2 and Theorem 5). E is obviously additive and homogeneous. If $f \epsilon \mathfrak{M}$, $Ef = f$ by the above and the range of E is included in $\mathfrak{M}$, $E\Lambda = \mathfrak{M}$ and $E^2 = E$. Thus E is a projection of Λ on $\mathfrak{M}$. Hence $C(\mathfrak{M})$ is $<\infty$.

Our construction also permits us to show that no statement concerning the norms of the ϕ_i and ψ_i will insure convergence by itself. We can assume that $\|\phi_i\| = 1$ for every i. The least possible value for $\|\psi_i\|$ is then 1 since $(\phi_i, \psi_i) = 1$ and $|(\phi_i, \psi_i)| \leq \|\phi_i\| \cdot \|\psi_i\|$. We will show that *there exists in both* l_p *and* L_p *a biorthogonal set* $\{\phi_i, \psi_i\}$ *for which* $\|\phi_i\| = \|\psi_i\| = 1$ *and for which the associated expansion* (*) *does not always converge.*

It is a consequence of the proof of Lemma 1.3.1 that if such a set exists in l_p, there must be a similar one in L_p. So we need only consider l_p. Owing to the nature of biorthogonal sets in l_p, we need only consider the case $1 < p < 2$.

Let $\mathfrak{M}$ be the manifold of §3.6, and let $f_1=(2^{-1/p}, 2^{-1/p}, 0)=\phi_1, f_2=(\alpha, -\alpha, -2\alpha)=h_1$. We have $(f_1, \{f_2\}^{p-1})=(\{f_1\}^{p-1}, f_2)=0$; $(f_1, \{f_1\}^{p-1})=(f_2, \{f_2\}^{p-1})=1$, $\|f_1\|=\|f_2\|=\|\{f_1\}^{p-1}\|=\|\{f_2\}^{p-1}\|=1$. Of course f_1 and $f_2 \epsilon l_{p,3}$, $\{f_1\}^{p-1}$ and $\{f_2\}^{p-1}\epsilon l_{p',3}$.

We define $f_{i_1}\otimes f_{i_2}\otimes \cdots \otimes f_{i_n}$, $i_j=1, 2; j=1, \cdots, n$, as an element of $l_{p,3^n}$ as follows: $f_{i_1}\otimes f_{i_2}$ in $l_{p,3^2}$ has already been defined (cf. §3.1). Let us suppose that $f_{i_1}\otimes \cdots \otimes f_{i_{n-1}}$ in $l_{p,3^{n-1}}$ has been defined. We define $f_{i_1}\otimes f_{i_2}\otimes \cdots \otimes f_{i_{n-1}}\otimes f_{i_n}$ as $(f_{i_1}\otimes \cdots \otimes f_{i_{n-1}})\otimes f_{i_n}$ in $l_{p,3^n}$ using §3.1. Let $\mathfrak{M}_1=\mathfrak{M}$, $\mathfrak{M}_n$ in $l_{p,3^n}$ be $\mathfrak{M}_{n-1}\otimes\mathfrak{M}$. Then by successive applications of Lemma 3.2.1, we see that the set of elements $f_{i_1}\otimes \cdots \otimes f_{i_n}$ determine $\mathfrak{M}_n$.

By Lemma 1.1.2, $l_p=\sum_{=1}^{\infty}\oplus l_{p,3^\alpha}$. Let $\mathfrak{P}$ be the closed linear manifold in $\sum_{=1}^{\infty}\oplus l_{p,3^\alpha}$ consisting of those elements $\{g_1, g_2, \cdots\}$ for which g_α is $\epsilon\mathfrak{M}_\alpha$ for every α. Let S consist of those elements which are such that every $g_\alpha=0$ except for one g_n and $g_n=f_{i_1}\otimes \cdots \otimes f_{i_n}$. Let S' consist of elements of $l_{p'}$ in the form $\{g\}^{p-1}$, $g\epsilon S$. Since as we have seen above the $f_{i_1}\otimes \cdots \otimes f_{i_n}$ determine $\mathfrak{M}_n$, S determines $\mathfrak{P}$.

Now the sets S and S' are denumerable and it is easily seen by using Lemma 3.1.1 that with a suitable enumeration they form a biorthogonal set with $\|\phi_i\|=\|\psi_i\|=1$. Since S determines $\mathfrak{P}$, we see from the above that this series (*) cannot converge always if $C(\mathfrak{P})=\infty$.

Let C be as in §3.6, (c). By repeated applications of Lemma 3.5.1 and then using Lemma 2.4.1, one may prove that $C(\mathfrak{M}_n)\geqq C^n$. It follows from the proof of Lemma 1.2.3 that $C(\mathfrak{P})$ is $\geqq C(\mathfrak{M}_n)\geqq C^n$ for every n and since C is >1, $p\neq 2$ this implies that $C(\mathfrak{P})=\infty$. As we have remarked above, this proves our statement.

Incidently we have explicitly constructed a manifold $\mathfrak{P}$ in l_p, for which there exists no projection. Lemma 1.3.1 indicates how we can find a $\mathfrak{P}$ in L_p with the corresponding property.

In L_p, the space of complex-valued functions whose pth power is summable, the situation is the same. As pointed out in a previous paper by the writer,† the theorems given in (B) and used here can be generalized to the complex case. Chapter 1 of this paper also falls into this category. Some variations are necessary in Chapters 2 and 3 but they are not basic.

Finally it should be pointed out that the negative answer to (a) and (b) precludes the possibility of a spectral theory in l_p and L_p similar to the theory of self-adjoint operators in Hilbert space.

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